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# Finite Grand Unified Theories and The Quark Mixing Matrix

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## Abstract

In super Yang–Mills theories, under certain conditions satisfied by the spectrum and the Yukawa couplings, the beta functions will vanish to all orders in perturbation theory. We address the generation of realistic quark mixing angles and masses in such finite Grand Unified Theories. Working in the context of finite SUSY  $SU(5)$ , we present several examples with realistic quark mixing matrices. Non-Abelian discrete symmetries are found to play a crucial role in satisfying the conditions for finiteness. Our realistic examples are based on permutation symmetries and the tetrahedral symmetry  $A_4$ . These examples enable us to address more detailed questions such as the decay rate of the proton in finite GUTs.

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# 1 Introduction

The Standard Model (SM) has confronted the experimental data with alarming success. Nevertheless, it is still considered by many as a low energy limit of a deeper underlying theory. The gauge hierarchy problem and the proliferation of parameters, especially in the fermion masses and mixings, are two of the major reasons for this belief. Various extensions of the SM address different aspects of these problems. Typically these extensions involve higher symmetries. While the number of effective parameters in these models with higher symmetries (e.g. grand unified theories) might be smaller than the SM, because of the necessity to break the higher symmetry, the actual number of parameters are often larger than the SM. Thus it becomes natural to ask whether it is possible to have a theory in which there are fewer number of parameters.

Indeed, there exists a certain class of supersymmetric Yang–Mills theories, where one may achieve this goal. These are the so-called finite theories wherein the  $\beta$  functions for the gauge coupling and the Yukawa couplings vanish to all orders in perturbation theory. Certain conditions must be satisfied for a SUSY Yang–Mills theory to be finite. One of them is the vanishing of the one-loop gauge  $\beta$  function. This requirement constrains the spectrum of the theory essentially fixing it (upto discrete possibilities). A second requirement for finiteness is the vanishing of all the anomalous mass dimensions of the chiral superfields at one-loop. This would fix all the Yukawa couplings in terms of the gauge coupling, at one-loop order. This type of one-loop finiteness also implies that the theory is finite to two loops [1]. For the theory to be finite to all loops, the Yukawa couplings must have unique power series expansions in the gauge coupling [2]. With this condition satisfied, the theory would have only one coupling – the gauge coupling. The Yukawa couplings are unified with the gauge couplings. This “reduction of couplings” is one of the key ingredients of finiteness [3, 4, 5, 6]. Certainly this makes the idea of finiteness an attractive direction to pursue in reducing the number of free parameters. One might hope that these type of theories may arise from superstring theory, which is known to be finite. Vanishing of the  $\beta$  functions lead to conformal invariance, which is one of the cornerstones of string theory. Indeed there have been several attempts to derive a grand unified theory from superstring theory as its low energy 4-D limit (for example see [7, 8]). The approach we adopt in this paper toward finiteness is that of [2]. Different approaches to finite theories can be found, for example, in [9, 10].

It will be extremely interesting to uncover finite theories that are phenomenologically viable, at least in a broad sense. Attempts have been made along this line with some success. An immediate question any finite theory should address is the consistency with the observed masses and mixings of the quarks. The Yukawa couplings are not arbitrary parameters in finite theories due to the reduction of couplings. The mass of the top-quark has been predicted within finite theories, in good agreement with experiments [11]. The masses of the lighter generation quarks have also been consistently accommodated in this context. However, the mixing between all three generations has not been implemented successfully thus far. This is the major point of the present paper.

We shall present three models based on finite SUSY  $SU(5)$  theory which can induce the correct pattern of quark mixing and masses. Additional flavor symmetries are necessary to

meet the criterion for finiteness that the power series expansion of the Yukawa couplings in terms of the gauge coupling be unique. We find that non-Abelian discrete symmetries are extremely useful here. Abelian symmetries that we have tried were not sufficient to make the expansion coefficients of the Yukawa couplings unique, non-Abelian continuous symmetries such as  $SU(2)$  and  $SU(3)$  are too restrictive to allow the needed Yukawa couplings. One of the examples that we present is based on permutation symmetry, the other is based on the tetrahedral symmetry  $A_4$ . We also present a third example based on  $S_4$  permutation symmetry in which the Yukawa couplings have a one-parameter family of solutions in terms of the gauge coupling. We anticipate that this arbitrariness may be removable by additional symmetries. If not, the proof of all-loop finiteness will not go through, although the theory will be finite to two-loop order. Such two-loop finite theories have been studied in Ref. [12]. It is not clear if the finiteness of the theory will be maintained by higher order corrections. Although these models with parametric solutions for the Yukawa couplings are more flexible (and thus less predictive) from the phenomenological point of view, we consider the all-loop finite models to be more attractive.

In Section 2, we review briefly the conditions for finiteness, starting from the renormalization group equations (RGE) for a generic supersymmetric theory. From one of the criteria, namely vanishing of the one loop gauge  $\beta$  function, it is not hard to see that finite models with phenomenologically favorable particle spectrum can be found more easily in  $SU(5)$  than in other groups [13]. Some general results of practical interest are given for finite  $SU(5)$  models. In Section 3 we propose three models and analyze them in detail. In all cases we show that realistic quark masses and mixing angles can be generated. This enables us to address more detailed questions such as the decay rate of the proton, which is perhaps one of the thorniest problems faced by SUSY GUTs. Generically finite theories are problematic [14], we give some plausible resolutions. Our conclusions are given in Section 4.

## 2 Finite Theories: A brief review

The one loop gauge beta function and the one loop anomalous dimension of the matter fields in a generic SUSY Yang-Mills theory are given by [1]:

$$\beta_g^{(1)} = \frac{1}{16\pi^2} \left( \sum_R T(R) - 3C_2(G) \right) \quad (1)$$

$$\gamma_j^{(1)i} = \lambda^{ikl} \lambda_{jkl} - 2C_2(R) g^2 \delta_j^i \quad (2)$$

$$\beta_{ijk}^{(1)} = \frac{1}{16\pi^2} [\lambda_{ijp} \gamma_k^p + (k \leftrightarrow i) + (k \leftrightarrow j)] \quad (3)$$

where  $T(R)$  and  $C_2(G)$  are the Dynkin indices of the matter fields and the quadratic Casimir of the gauge group respectively.  $\lambda^{ijk}$  and  $\beta_{ijk}^{(1)}$  are the Yukawa couplings and the one-loop Yukawa  $\beta$  function of  $\lambda^{ijk}$ . The criteria of all loop finiteness for  $N = 1$  supersymmetric gauge theories can be stated as follows [2]: (i) It should be free from gauge anomaly, (ii) the gauge  $\beta$ -function vanishes at one loop:

$$\beta_g^{(1)} = 0, \quad (4)$$

(iii) there exists solution of the form  $\lambda = \lambda(g)$  to the conditions of vanishing one-loop anomalous dimensions

$$\gamma_j^{(1)i} = 0, \quad (5)$$

and (iv) the solution is isolated and non-degenerate when considered as a solution of vanishing one-loop Yukawa  $\beta$ -function:

$$\beta_{ijk}^{(1)} = 0. \quad (6)$$

If all conditions are satisfied, the theory would depend on a single gauge coupling constant and the  $\beta$  functions will vanish to all orders.

The first step is to choose the gauge group. From (i), we see that the vanishing of the one loop gauge  $\beta$  function puts a strong constraint on the particle content, leaving only discrete set of possibilities. It seems hard to find phenomenologically viable models other than in  $SU(5)$  [13]. For example, if one chooses  $SO(10)$  to be the gauge group, and tries to build a finite model with necessary particle content in the traditional sense, one quickly “runs” out of the Dynkin indices: according to Eq. (1) the sum of the Dynkin indices over the matter fields should be equal to 24 in this case. On the other hand, field content of the traditional  $SO(10)$  GUT is  $3 \times \mathbf{16}$  of fermions,  $\mathbf{54}$ ,  $\mathbf{45}$ ,  $\mathbf{10} + \mathbf{\overline{10}}$  and  $\mathbf{16} + \mathbf{\overline{16}}$  of Higgs, if the gauge symmetry is to be broken by renormalizable terms in the superpotential. Then, since the sum of the Dynkin indices of these fields is equal to 32, one ends up exceeding the gauge Dynkin index. Much the same result can be reached for  $SU(6)$  etc. While it will be of great interest to uncover finite models other than  $SU(5)$ , here we will confine ourselves to the case of finite  $SU(5)$ .

Beginning with the particle content of minimal SUSY  $SU(5)$  theory with three families of fermions belonging to  $3 \times (\mathbf{10} + \mathbf{\overline{5}})$ , an adjoint  $\mathbf{24}$  Higgs ( $\Sigma$ ) and  $(\mathbf{5} + \mathbf{\overline{5}})$  Higgses one sees that vanishing of the one-loop gauge  $\beta$  function requires the introduction of additional fields whose Dynkin indices add up to 3. This happens if there are three additional  $\mathbf{5} + \mathbf{\overline{5}}$  matter fields, which may be either Higgs-like bosonic fields or vector-like fermionic fields. This is in fact the most promising case from phenomenology. There are two other possibilities, viz., adding  $\mathbf{10} + \mathbf{\overline{10}}$  or adding  $\mathbf{10} + \mathbf{5} + 2 \times \mathbf{\overline{5}}$ . In the first case, realistic quark masses cannot arise, in the second case one would be left with a fourth family of fermions which remains light to the weak scale. For phenomenological reasons we do not pursue these two alternatives, and choose to work with  $3 \times \{\mathbf{5} + \mathbf{\overline{5}}\}$  plus the minimal SUSY  $SU(5)$  spectrum.

The finiteness criteria require that Eqs. (1)-(3) should give a unique set of solutions to the Yukawa couplings. The equations are linear in the square of the absolute values of the couplings. Hence one expects the solutions to the Yukawa couplings to be either zero or of order the gauge coupling. They are not free parameters anymore. In order to satisfy the hierarchy in masses of the fermions, one can choose the VEVs of the Higgs bosons to be hierarchical. Naively this would need at least three Higgs multiplets coupling to the up-quark sector, and three multiplets coupling to the down-quark sector. It is interesting that finite  $SU(5)$  spectrum admits the needed Higgs, which can be as many as 4 in each sector. We will be interested in the case where at least three of the  $\mathbf{5} + \mathbf{\overline{5}}$  fields are Higgs-like (viz., they develop VEVs of the order the electroweak scale). In fact, we shall see shortly that independent of this phenomenological requirement, the vanishing of the one-loop anomalous

dimensions necessitates that at least three pairs of  $\mathbf{5} + \bar{\mathbf{5}}$  have Yukawa couplings to the three families of fermions. We will focus on inducing realistic mixing among the three families of quarks, which has not been achieved in earlier analyses [11].

To search for a finite model, one has to write down a specific superpotential and try to find a set of solutions satisfying the criteria that all the Yukawa coupling wave function renormalization factors vanish at one-loop. We consider the following superpotential (assuming an unbroken  $R$ -parity):

$$\begin{aligned}
W = & \sum_{i,j=1}^3 \sum_a \left( \frac{1}{2} u_{ij}^a \mathbf{10}_i \mathbf{10}_j H_a + d_{ij}^a \mathbf{10}_i \bar{\mathbf{5}}_j \bar{H}_a \right) \\
& + \sum_{ab} k^{ab} H_a \Sigma \bar{H}_b + \frac{\lambda}{3} \Sigma^3 + f \mathbf{5} \Sigma \bar{\mathbf{5}}.
\end{aligned} \tag{7}$$

Here  $i, j = (1 - 3)$  are family indices and  $a$  and  $b$  are Higgs indices.  $a$  and  $b$  run from 1 to either 3 or 4. If it is up to 4, the last term is absent.  $H$  and  $\bar{H}$  denote the  $\mathbf{5} + \bar{\mathbf{5}}$  fields and  $\Sigma$  the adjoint chiral matter field responsible for the GUT symmetry breaking. Note that in order to have a successful doublet-triplet mass splitting, at least one of the couplings  $f, k^{ab}$  should be non-vanishing.

From Eq. (7), the anomalous dimensions of Eq. (2) can be written in matrix form as:

$$\begin{aligned}
\gamma_{\mathbf{10}_i \mathbf{10}_j} &= 3(u_a u_a^\dagger)_{ij} + 2(d_a d_a^\dagger)_{ij} - \frac{36}{5} g^2 \delta_{ij} \\
\gamma_{\bar{\mathbf{5}}_i \bar{\mathbf{5}}_j} &= 4(d_a^\dagger d_a)_{ij} - \frac{24}{5} g^2 \delta_{ij} \\
\gamma_{H_a H_b} &= 3Tr(u_a^\dagger u_b) + \frac{24}{5} (k k^\dagger)_{ab} - \frac{24}{5} g^2 \delta_{ab} \\
\gamma_{\bar{H}_a \bar{H}_b} &= 4Tr(d_a d_b^\dagger) + \frac{24}{5} (k^\dagger k)_{ab} - \frac{24}{5} g^2 \delta_{ab} \\
\gamma_{\mathbf{5}} &= \gamma_{\bar{\mathbf{5}}} = \frac{24}{5} f^2 - \frac{24}{5} g^2 \\
\gamma_{24} &= Tr(k^\dagger k) + f^2 + \frac{21}{5} \lambda^2 - 10g^2.
\end{aligned} \tag{8}$$

According to the third criteria, Eq. (5), in order to have finite theory, all these anomalous mass dimension have to be zero. Thus, the problem of finding a finite model shifts to the problem of finding a set of solutions, where all the anomalous dimensions in Eq. (8) vanish. Let us introduce a new notations for the matrices:

$$\begin{aligned}
U \equiv u_a u_a^\dagger, \quad D \equiv d_a d_a^\dagger, \quad D' \equiv d_a^\dagger d_a, \quad \tilde{U}_{ab} \equiv Tr(u_a^\dagger u_b), \\
\tilde{D}_{ab} \equiv Tr(d_a d_b^\dagger), \quad K \equiv k^\dagger k, \quad \tilde{K} \equiv k k^\dagger.
\end{aligned} \tag{9}$$

From Eq. (8), it follows that the number of  $H$  fields coupling to  $\mathbf{10}_i \mathbf{10}_j$  should be equal to the number of  $\bar{H}$  fields coupling to  $\mathbf{10}_i \bar{\mathbf{5}}_j$  fields. Furthermore, at least 3  $H$  fields (and 3  $\bar{H}$  fields) must have such couplings. To see this let us take the trace of the matrices of the

anomalous dimensions in Eq. (8) over both the fermionic indices and the Higgs indices. One gets:

$$\begin{aligned}
3\text{Tr}(U) + 2\text{Tr}(D) &= 3 \times \frac{36}{5}g^2 \\
4\text{Tr}(D') &= 3 \times \frac{24}{5}g^2 \\
3\text{Tr}(\tilde{U}) + \frac{24}{5}\text{Tr}(\tilde{K}) &= n_H \times \frac{24}{5}g^2 \\
4\text{Tr}(\tilde{D}) + \frac{24}{5}\text{Tr}(K) &= n_{\bar{H}} \times \frac{24}{5}g^2,
\end{aligned} \tag{10}$$

where  $n_H$  and  $n_{\bar{H}}$  are the number of the Higgs fields coupling to the three family of fermions in the up-sector and the down-sector respectively. Subtracting the third equation from the last in Eq. (10), we get

$$4\text{Tr}(\tilde{D}) - 3\text{Tr}(\tilde{U}) = (n_{\bar{H}} - n_H) \times \frac{24}{5}g^2.$$

Observing the following relation

$$\text{Tr}(U) = \text{Tr}(\tilde{U}) \quad \text{Tr}(D) = \text{Tr}(D') = \text{Tr}(\tilde{D}),$$

one finds that

$$n_H = n_{\bar{H}}. \tag{11}$$

One can also see that the matrices  $K$  and  $\tilde{K}$  in Eq. (9) vanish if  $n_H = n_{\bar{H}} = 3$ . That is,  $k_{ab} = 0$  for all  $(a, b)$ . Doublet-triplet splitting can be achieved in this case since  $f \neq 0$  is allowed. If  $n_H = n_{\bar{H}} \leq 2$ , no solution exists for Eq. (10). We conclude that at least three Higgs multiplets must couple to the fermion fields in finite  $SU(5)$ .

Vanishing of the right-hand side of Eq. (8), needed for finiteness, will in general lead to parametric solutions. In order to satisfy the condition for all-loop finiteness, additional symmetries are usually necessary. Under these extra symmetries different Higgs multiplets will have different charges, which would prevent them from coupling to the same set of fermion fields. If two different Higgs multiplets  $H_1$  and  $H_2$  coupled to the same fermion fields, say  $\mathbf{10}_1 \mathbf{10}_2$ , then  $\gamma_{H_1 H_2}$  will not vanish in general, and so the theory will not be finite. We now present a classification of the Yukawa coupling matrices which ensures in a simple way that the off-diagonal entries of the anomalous dimension matrices are all automatically zero. While this classification is not the most general, it can be applied to a wide class of models. Let us write the superpotential Eq. (7) in the following form:

$$W = \frac{1}{2} \mathbf{10}_i \mathbf{10}_j V^u_{ij} + \mathbf{10}_i \bar{\mathbf{5}}_j V^d_{ij} + \dots, \tag{12}$$

where

$$V^u_{ij} = u^a_{ij} H_a, \quad V^d_{ij} = d^a_{ij} \bar{H}_a.$$

The structures of  $V^u$  matrices which automatically have all off-diagonal anomalous dimensions to be zero is obtained as follows. Consider the case where three pairs of  $(H, \bar{H})$  couple to the chiral families. There are four distinct forms for the matrix  $V^u$ :

$$\begin{aligned}
V^{(1)} &\equiv \begin{pmatrix} u_{11}H_1 & u_{12}H_3 & u_{13}H_2 \\ u_{12}H_3 & u_{22}H_2 & u_{23}H_1 \\ u_{13}H_2 & u_{23}H_1 & u_{33}H_3 \end{pmatrix} & V^{(2)} &\equiv \begin{pmatrix} u_{11}H_2 & u_{12}H_1 & 0 \\ u_{12}H_1 & u_{22}H_3 & u_{23}H_2 \\ 0 & u_{23}H_2 & u_{33}H_3 \end{pmatrix} \\
V^{(3)} &\equiv \begin{pmatrix} u_{11}H_3 & u_{12}H_1 & 0 \\ u_{12}H_1 & u_{22}H_3 & u_{23}H_2 \\ 0 & u_{23}H_2 & u_{33}H_3 \end{pmatrix} & V^{(4)} &\equiv \begin{pmatrix} u_{11}H_1 & 0 & 0 \\ 0 & u_{22}H_3 & u_{23}H_2 \\ 0 & u_{23}H_2 & u_{33}H_3 \end{pmatrix}. \quad (13)
\end{aligned}$$

The form of  $V^d$  in this case is identical to Eq. (13), except that  $u_{ij}$  is replaced by  $d_{ij}$  and  $H_i$  by  $\bar{H}_i$ . While  $V^u$  is a symmetric matrix,  $V^d$  is asymmetric. Any given Higgs field appears at most once in a given row or column in all the matrices of Eq. (13). This guarantees that all off-diagonal  $\gamma$  function entries are zero. It can be shown that Eq. (13) is the most general set of matrices that satisfy this constraint (upto relabeling of generation number and Higgs number), provided that there is no cancellation between various terms to generate a zero in the off-diagonal  $\gamma$  matrix. It is possible that such cancellations occur in the presence of non-Abelian flavor symmetries, but not with Abelian symmetries. Even for the case of non-Abelian symmetries, we have found the classification of Eq. (13) very useful.

If four pairs of  $(H + \bar{H})$  couple to fermion families, the matrix  $V^u$  can have the following four structures (upto relabeling of generation and Higgs indices):

$$\begin{aligned}
V^{(1)} &\equiv \begin{pmatrix} u_{11}H_1 & u_{12}H_4 & u_{13}H_2 \\ u_{12}H_4 & u_{22}H_2 & u_{23}H_1 \\ u_{13}H_2 & u_{23}H_1 & u_{33}H_3 \end{pmatrix} & V^{(2)} &\equiv \begin{pmatrix} u_{11}H_2 & u_{12}H_1 & 0 \\ u_{12}H_1 & u_{22}H_2 & u_{23}H_4 \\ 0 & u_{23}H_4 & u_{33}H_3 \end{pmatrix} \\
V^{(3)} &\equiv \begin{pmatrix} u_{11}H_3 & u_{12}H_1 & u_{13}H_2 \\ u_{12}H_1 & u_{22}H_2 & u_{23}H_4 \\ u_{13}H_2 & u_{23}H_4 & u_{33}H_3 \end{pmatrix} & V^{(4)} &\equiv \begin{pmatrix} u_{11}H_3 & u_{12}H_1 & u_{13}H_2 \\ u_{12}H_1 & u_{22}H_3 & u_{23}H_4 \\ u_{13}H_2 & u_{23}H_4 & u_{33}H_3 \end{pmatrix}. \quad (14)
\end{aligned}$$

$V^d$  in this case will have similar structure, assuming that its form is similar to  $V^u$ . In all cases, one can easily verify that the off-diagonal contributions to the anomalous dimension matrices are all zero.

### 3 The Quark Mixing in Finite GUT

It is possible to find solutions for the vanishing of the anomalous dimensions of Eq. (8) with the forms of  $V^u$  and  $V^d$  given as in Eq. (13)-(14). We have examined all possible cases, including  $V^u$  taking the form of  $V^{(i)}$  while  $V^d$  taking the form of  $V^{(j)}$  with  $i$  and  $j$  not necessarily the same. We found parametric solutions wherein one or (typically) more

parameters are not determined. That would forbid a unique expansion of the Yukawa couplings in terms of the gauge coupling, one of the requirements of finiteness. It is possible to remove this arbitrariness by imposing additional flavor symmetries. Three examples of this type are proposed here and analyzed in detail. In the first example, isolated non-degenerate solution to the Yukawa couplings is obtained by imposing a  $(Z_4)^3 \times P$  symmetry, where  $P$  stands for permutation. The second example is based on the tetrahedral group  $A_4$ . A third example based on  $S_4$  symmetry is also presented, which actually gives a one parameter family of solutions. If this parameter is chosen to have a specific value (which we believe can be enforced by a symmetry) the solutions will again be isolated and non-degenerate.

### 3.1 $(Z_4)^3 \times P$ Model

Let us give the transformation properties of the fields under the discrete symmetry we impose. The symmetries are  $(Z_4)^3$ , identified as the  $Z_4$  subgroup of generation number, and a permutation symmetry acting on both the fermion and the Higgs generations. The fields transform under  $(Z_4)^3$  as:

$$\begin{aligned} \mathbf{10}_1 &: (i, 1, 1), & \mathbf{10}_2 &: (1, i, 1), & \mathbf{10}_3 &: (1, 1, i), \\ \bar{\mathbf{5}}_1 &: (i, 1, 1), & \bar{\mathbf{5}}_2 &: (1, i, 1), & \bar{\mathbf{5}}_3 &: (1, 1, i), \\ (H_1, \bar{H}_1) &: (-1, 1, 1), & (H_2, \bar{H}_2) &: (1, -i, -i), \\ (H_3, \bar{H}_3) &: (1, 1, -1), & (H_4, \bar{H}_4) &: (1, -i, -i). \end{aligned} \tag{15}$$

The action of the permutation symmetry  $P$  on the fields is as follows:

$$\begin{aligned} \mathbf{10}_1 &\leftrightarrow \mathbf{10}_3, & \bar{\mathbf{5}}_1 &\leftrightarrow \bar{\mathbf{5}}_3, & H_1 &\leftrightarrow H_3, & \bar{H}_1 &\leftrightarrow \bar{H}_3, \\ \mathbf{10}_2 &\leftrightarrow \mathbf{10}_2, & \bar{\mathbf{5}}_2 &\leftrightarrow \bar{\mathbf{5}}_2, & H_2 &\leftrightarrow H_4, & \bar{H}_2 &\leftrightarrow \bar{H}_4. \end{aligned}$$

The most general  $SU(5) \times (Z_4)^3 \times P$  invariant superpotential is:

$$\begin{aligned} W &= a(\mathbf{10}_1 \mathbf{10}_1 H_1 + \mathbf{10}_3 \mathbf{10}_3 H_3) + b(\mathbf{10}_1 \mathbf{10}_2 H_4 + \mathbf{10}_2 \mathbf{10}_3 H_2) \\ &+ c(\mathbf{10}_1 \bar{\mathbf{5}}_1 \bar{H}_1 + \mathbf{10}_3 \bar{\mathbf{5}}_3 \bar{H}_3) + d(\mathbf{10}_1 \bar{\mathbf{5}}_2 \bar{H}_4 + \mathbf{10}_3 \bar{\mathbf{5}}_2 \bar{H}_2) \\ &+ e(\mathbf{10}_2 \bar{\mathbf{5}}_1 \bar{H}_4 + \mathbf{10}_2 \bar{\mathbf{5}}_3 \bar{H}_2) + k(H_1 \bar{H}_1 \Sigma + H_3 \bar{H}_3 \Sigma) + \frac{\lambda}{3} \Sigma^3. \end{aligned} \tag{16}$$

The matrices  $V^u$  and  $V^d$  (defined in Eq. (14)) for this model are then:

$$V^u = \begin{pmatrix} a H_1 & b H_4 & 0 \\ b H_4 & 0 & b H_2 \\ 0 & b H_2 & a H_3 \end{pmatrix} \quad V^d = \begin{pmatrix} c \bar{H}_1 & d \bar{H}_4 & 0 \\ e \bar{H}_4 & 0 & e \bar{H}_2 \\ 0 & d \bar{H}_2 & c \bar{H}_3 \end{pmatrix}, \tag{17}$$

and the coupling matrix of the Higgs fields to the adjoint field is given by:

$$K = \text{diag}(k, 0, k, 0).$$



Note that all superpotential couplings can be made real by field redefinitions. One can then take all parameters of Eq. (16) to be real and positive. This is an important point for the solution to be non-degenerate.

The condition (iii) of the criteria for finiteness (vanishing of all the anomalous dimensions) leads to the following simple system of equations:

$$\begin{aligned} 3(a^2 + b^2) + 2(c^2 + d^2) &= \frac{36}{5}g^2, & 3(2b^2) + 2(2e^2) &= \frac{36}{5}g^2, \\ c^2 + e^2 &= \frac{6}{5}g^2, & 2b^2 &= \frac{8}{5}g^2, \\ a^2 + \frac{8}{5}k^2 &= \frac{8}{5}g^2, & d^2 + e^2 &= \frac{6}{5}g^2, \\ 2d^2 &= \frac{6}{5}g^2, & a^2 + \frac{6}{5}k^2 &= \frac{6}{5}g^2. \end{aligned} \quad (18)$$

This gives a unique solution which is isolated and non-degenerate:

$$(a^2, b^2, c^2, d^2, e^2, k^2, \lambda^2) = \left( \frac{4}{5}g^2, \frac{4}{5}g^2, \frac{3}{5}g^2, \frac{3}{5}g^2, \frac{1}{2}g^2, \frac{15}{7}g^2 \right). \quad (19)$$

There is no sign ambiguity for the Yukawa couplings themselves, since they have all been made real and positive.

Let us now turn to the question of comparing the predictions of Eq. (19) with experiments. First of all, all three families of quarks mix with one another, so realistic CKM mixings become possible, unlike earlier attempts within finite GUTs. Setting the overall factor  $a\langle H_3 \rangle = 1$ , we can write the mass matrix  $M^u$  for the up-type quarks in the following form:

$$M^u = \begin{pmatrix} c_{11}^u \epsilon_u^4 & c_{12}^u \epsilon_u^3 & 0 \\ c_{21}^u \epsilon_u^3 & 0 & \epsilon_u \\ 0 & \epsilon_u & 1 \end{pmatrix}, \quad (20)$$

where  $\epsilon_u \equiv \frac{\langle H_2 \rangle}{\langle H_3 \rangle}$ ,  $c_{11}^u \epsilon_u^4 \equiv \frac{\langle H_1 \rangle}{\langle H_3 \rangle}$  and  $c_{12}^u \epsilon_u^3 \equiv \frac{\langle H_4 \rangle}{\langle H_3 \rangle}$ . The mass matrix for the down-type quarks,  $M^d$ , has a similar form, with  $\epsilon_u$  replaced by  $\epsilon_d$  and  $c_{ij}^u$  replaced by  $c_{ij}^d$ . These matrices are generalizations of the Fritzsch form. Note that Eq. (20) is a special case of texture  $V^{(2)}$  in Eq. (14). The mass eigenvalues are obtained in the approximation  $\epsilon_u \ll 1$ ,  $c_{ij}^e \sim 1$  to be:

$$\begin{aligned} m_u &\simeq (c_{11}^u + (c_{12}^u)^2) \epsilon_u^4 \\ m_c &\simeq -\epsilon_u^2 + (1 - c_{12}^u)^2 \epsilon_u^4 \\ m_t &\simeq 1 + \epsilon_u^2 - \epsilon_u^4 \end{aligned} \quad (21)$$

in units of  $a\langle H_3 \rangle$ . Similar expressions hold in the down-type quark sector. The CKM mixing elements are then given by:

$$\begin{aligned} V_{us} &= c_{12}^u \epsilon_u - c_{12}^d \epsilon_d + O(\epsilon^3) \\ V_{cb} &= \epsilon_d - \epsilon_u + O(\epsilon^3) \\ V_{ub} &= c_{12}^u \epsilon_u \epsilon_d - c_{12}^u \epsilon_u^2 + O(\epsilon^4), \end{aligned} \quad (22)$$

where  $\epsilon_d$  and  $c_{12}^d$  correspond to the down quark sector. (For simplicity, we have assumed all parameters to be real. This assumption is not necessary, realistic CP violation can also arise from Eq. (20).)

Observe that the mass hierarchy between generations can be accommodated in this model by assuming a hierarchy in the VEVs of the Higgs doublets. We have in mind a scenario where only one pair of Higgs doublets survive below the GUT scale, to be identified as  $H_u$  and  $H_d$  of MSSM. These are linear combinations of all four of the original Higgs doublets. That would enable all  $H_i$  ( $i = 1 - 4$ ) to acquire VEVs. The  $H_u$  field of MSSM is dominantly  $H_3$ , but has small (of order  $\epsilon_u$ ) component of  $H_2$  in it, and even smaller components of  $H_3$  (of order  $\epsilon_u^3$ ) and  $H_4$  (of order  $\epsilon_u^4$ ) in it. These amounts are dictated by the bilinear terms in the superpotential involving  $H_i$  and  $\bar{H}_i$  fields ( $W \sim m_{ij} H_i \bar{H}_j$ ). These bilinear terms are assumed to break the  $(Z_4)^3 \times P$  symmetry softly. We see that the desired mass hierarchy is reproduced in this way.

Since the Yukawa couplings of the third generation quarks are fixed in this model in terms of the gauge coupling, the mass of the top quark and the parameter  $\tan \beta$  are determined. Let us denote the MSSM Yukawa couplings of the top and the bottom quarks to  $H_u$  and  $H_d$  fields to be  $y_t$  and  $y_b$  respectively. To a good approximation,  $H_u$  is  $H_3$  and  $H_d$  is  $\bar{H}_3$ . Thus we see from Eq. (19) that  $y_t \simeq (\sqrt{4/5})g$  and  $y_b \simeq (\sqrt{3/5})g$ , both of which are fixed in terms of  $\alpha_G \simeq 1/25$ . We now extrapolate these Yukawa couplings to the weak scale using the MSSM renormalization group equations. The top quark mass and the parameter  $\tan \beta$  can be predicted using the relations

$$\begin{aligned} m_t &= m_b \frac{y_t}{y_b} \sqrt{y_b^2 \frac{v^2}{m_b^2} - 1} \simeq y_t v \\ \tan \beta &= \frac{m_t y_b}{m_b y_t}. \end{aligned} \quad (23)$$

With  $m_b(m_b)$  taken to be 4.4 GeV and with  $\alpha_3(m_Z) = 0.118$  we find the numerical values to be:

$$\begin{aligned} m_t &= 174 \text{ GeV} \\ \tan \beta &= 53. \end{aligned} \quad (24)$$

The predicted value of  $m_t$  is nicely consistent with the experimentally determined value,  $\tan \beta$  tends to be large in this class of models.

There is one other non-trivial prediction in this model, because of the zeros present in Eq. (20). We take it to be a prediction for the strange quark mass. From Eqs. (21) and (22) we find  $m_s(1 \text{ GeV}) \simeq 80 \text{ MeV}$ , if we take  $V_{cb} \simeq 0.043$ ,  $m_b(m_b) = 4.4 \text{ GeV}$ ,  $m_c(m_c) = 1.37 \text{ GeV}$ ,  $V_{us} = 0.22$  and  $V_{ub} = 0.004$ . This value of  $m_s$  is on the low side, but may be consistent with recent lattice evaluations [15]. We also note that since  $\tan \beta$  is predicted to be large, the finite threshold corrections to  $V_{cb}$  through chargino-stop exchange is significant [16]. This could modify  $V_{cb}$  by as much as 30%. With a 30% reduction in  $V_{cb}$  arising from this diagram, we predict  $m_s(1 \text{ GeV}) \simeq 100 \text{ MeV}$ , which is quite acceptable.

### 3.2 $A_4$ Model

Now we present a second model that leads to realistic quark mixings and masses. It is based on  $SU(5) \times A_4$  symmetry.  $A_4$  is the group of even permutations of four objects. It is the symmetry group of a regular tetrahedron. This group has irreducible representations (denoted by the dimensions) 1, 1', 1'' and 3. The 1' and 1'' are complex conjugate of each other. The product  $3 \times 3$  decomposes as

$$3 \times 3 = 1 + 1' + 1'' + 3 + 3. \quad (25)$$

If we denote the components of 3 as (a,b,c), the various terms are given by [17]:

$$\begin{aligned} 1 &= a_1 a_2 + b_1 b_2 + c_1 c_2 \\ 1' &= a_1 a_2 + \omega^2 b_1 b_2 + \omega c_1 c_2 \\ 1'' &= a_1 a_2 + \omega b_1 b_2 + \omega^2 c_1 c_2, \end{aligned} \quad (26)$$

where  $\omega = e^{2i\pi/3}$ . (Note that  $1 + \omega + \omega^2 = 0$ .)

The transformations properties of the fields of  $SU(5)$  under  $A_4$  are:

$$\begin{aligned} \mathbf{10}_i &: 3 & \bar{\mathbf{5}}_i &: 1 + 1' + 1'' \\ (H_a, H_4) &: 3 + 1' & (\bar{H}_a, H_4) &: 3 + 1' \quad \Sigma = 1, \end{aligned} \quad (27)$$

where  $i = 1 \div 3$  and  $a = 1 \div 3$ . Using the algebra presented in Eq. (26), the superpotential invariant under  $SU(5) \times A_4$  symmetry is:

$$\begin{aligned} W &= a(\mathbf{10}_1 \mathbf{10}_1 + \omega^2 \mathbf{10}_2 \mathbf{10}_2 + \omega \mathbf{10}_3 \mathbf{10}_3) H_4 \\ &+ c(\mathbf{10}_1 \bar{\mathbf{5}}_1 + \omega^2 \mathbf{10}_2 \bar{\mathbf{5}}_2 + \omega \mathbf{10}_3 \bar{\mathbf{5}}_3) \bar{H}_4 \\ &+ b(\mathbf{10}_1 \mathbf{10}_2 + \mathbf{10}_2 \mathbf{10}_1) H_1 + d(\mathbf{10}_1 \bar{\mathbf{5}}_2 + \mathbf{10}_2 \bar{\mathbf{5}}_1) \bar{H}_1 \\ &+ b(\mathbf{10}_1 \mathbf{10}_3 + \mathbf{10}_3 \mathbf{10}_1) H_2 + d(\mathbf{10}_1 \bar{\mathbf{5}}_3 + \mathbf{10}_3 \bar{\mathbf{5}}_1) \bar{H}_2 \\ &+ \omega b(\mathbf{10}_2 \mathbf{10}_3 + \mathbf{10}_3 \mathbf{10}_2) H_3 + \omega c(\mathbf{10}_2 \bar{\mathbf{5}}_3 + \mathbf{10}_3 \bar{\mathbf{5}}_2) \bar{H}_3 \\ &+ k(\bar{H}_1 H_1 + \bar{H}_2 H_2 + \bar{H}_3 H_3) + \frac{\lambda}{3} \Sigma^3. \end{aligned} \quad (28)$$

By field redefinition the  $\omega$  factors can be removed from  $W$ . Actually, all the coupling constants in Eq. (28) can be made real and positive. The condition of vanishing anomalous dimensions for this model can be written as follows:

$$\begin{aligned} 3(a^2 + 2b^2) + 2(c^2 + 2d^2) &= \frac{36}{5} g^2 \\ 4(c^2 + 2d^2) &= \frac{24}{5} g^2 \\ 3(3a^2) &= \frac{24}{5} g^2 \\ 3(2b^2) + \frac{24}{5} k^2 &= \frac{24}{5} g^2 \\ 4(3c^2) &= \frac{24}{5} g^2 \\ 4(2d^2) + \frac{24}{5} k^2 &= \frac{24}{5} g^2. \end{aligned} \quad (29)$$

This gives the following isolated and non-degenerate solution:

$$a^2 = b^2 = \frac{8}{15}g^2, \quad c^2 = d^2 = e^2 = \frac{2}{5}g^2, \quad k^2 = \frac{1}{3}g^2. \quad (30)$$

The resulting up-quark mass matrix can be written as:

$$M^u = \sqrt{\frac{8}{15}}g\langle H_4 \rangle \begin{pmatrix} 1 & 1 + \epsilon_1 & 1 + \epsilon_2 \\ 1 + \epsilon_1 & 1 & 1 + \epsilon_3 \\ 1 + \epsilon_2 & 1 + \epsilon_3 & 1 \end{pmatrix}, \quad (31)$$

where

$$\epsilon_1 = \frac{\langle H_1 \rangle}{\langle H_4 \rangle} - 1, \quad \epsilon_2 = \frac{\langle H_1 \rangle}{\langle H_4 \rangle} - 1, \quad \epsilon_3 = \frac{\langle H_1 \rangle}{\langle H_4 \rangle} - 1, \quad (32)$$

with a similar form for the down-type quark matrix. We can accommodate the mass hierarchy by taking  $\epsilon_{1,2,3} \ll 1$ . This structure has been considered in [18], where it has been shown to agree well with experimental data.

In the  $A_4$  model, since all  $H_i$  have almost equal VEVs,  $\langle H_4 \rangle \simeq \langle H_u \rangle / 2$ . Furthermore, from Eq. (31), we have  $m_t \simeq 3\sqrt{8/15}g\langle H_4 \rangle$ , so that  $y_t = (\sqrt{6/5})g$  at the GUT scale. Similarly,  $y_b = (\sqrt{9/10})g$  at the GUT scale. These boundary conditions lead to the predictions

$$\begin{aligned} m_t &= 177 \text{ GeV} \\ \tan \beta &= 53. \end{aligned} \quad (33)$$

As shown in Ref. ([18]), all the quark mixing angles can be correctly reproduced in this model.

### 3.3 $S_4$ Model

We now present a third example based on  $S_4$  symmetry. This symmetry alone would lead to a one parameter family of solutions for the Yukawa couplings. Although we have not found a symmetry that will uniquely fix this parameter, we suspect that such a symmetry might actually exist. Keeping this in mind, we proceed to analyze this model.  $S_4$  is the permutation symmetry operating on four objects. It has the following irreducible representations: (1, 1', 2, 3, 3') [19]. We choose the following assignment of the chiral superfields under  $S_4$ :

$$\begin{aligned} \mathbf{10}_i &: 3, & (H_a, H_4) &: 3 + 1, & \Sigma &: 1, \\ \bar{\mathbf{5}}_i &: 3, & (\bar{H}_a, \bar{H}_4) &: 3 + 1, \end{aligned} \quad (34)$$

The superpotential invariant under this symmetry is

$$\begin{aligned} W &= a[(\mathbf{10}_1\mathbf{10}_3 + \mathbf{10}_3\mathbf{10}_1)H_1 + (\mathbf{10}_2\mathbf{10}_3 + \mathbf{10}_3\mathbf{10}_2)H_2 + (\mathbf{10}_1\mathbf{10}_1 - \mathbf{10}_2\mathbf{10}_2)H_1] \\ &+ b(\mathbf{10}_1\mathbf{10}_1 + \mathbf{10}_2\mathbf{10}_2 + \mathbf{10}_3\mathbf{10}_3)H_4 \\ &+ c[(\mathbf{10}_1\bar{\mathbf{5}}_3 + \mathbf{10}_3\bar{\mathbf{5}}_1)\bar{H}_1 + (\mathbf{10}_2\bar{\mathbf{5}}_3 + \mathbf{10}_3\bar{\mathbf{5}}_2)\bar{H}_2 + (\mathbf{10}_1\bar{\mathbf{5}}_1 - \mathbf{10}_2\bar{\mathbf{5}}_2)\bar{H}_4] \\ &+ d(\mathbf{10}_1\bar{\mathbf{5}}_1 + \mathbf{10}_2\bar{\mathbf{5}}_2 + \mathbf{10}_3\bar{\mathbf{5}}_3)\bar{H}_4 \\ &+ k[(H_1\bar{H}_1 + H_2\bar{H}_2 + H_3\bar{H}_3)\Sigma] \\ &+ k_4H_4\bar{H}_4\Sigma + \frac{\lambda}{3}\Sigma^3. \end{aligned} \quad (35)$$

The  $V^u$  and  $V^d$  matrices that arise from this superpotential are:

$$V^u = \begin{pmatrix} a\langle H_3 \rangle + b\langle H_4 \rangle & 0 & a\langle H_1 \rangle \\ 0 & b\langle H_4 \rangle & a\langle H_2 \rangle \\ a\langle H_1 \rangle & a\langle H_2 \rangle & -a\langle H_3 \rangle + b\langle H_4 \rangle \end{pmatrix}, \quad (36)$$

$$V^d = \begin{pmatrix} c\langle \bar{H}_3 \rangle + d\langle \bar{H}_4 \rangle & 0 & c\langle \bar{H}_1 \rangle \\ 0 & d\langle \bar{H}_4 \rangle & c\langle \bar{H}_2 \rangle \\ c\langle \bar{H}_1 \rangle & c\langle \bar{H}_2 \rangle & -c\langle \bar{H}_3 \rangle + d\langle \bar{H}_4 \rangle \end{pmatrix}. \quad (37)$$

The coupling matrix  $k$  connecting the Higgs fields  $(H, \bar{H})$  and the adjoint field  $\Sigma$  is:

$$k = \text{diag}(k, k, k, k_4).$$

The condition for vanishing anomalous mass dimensions is then:

$$\begin{aligned} 3(2a^2 + b^2) + 2(2c^2 + d^2) &= \frac{36}{5}g^2 \\ 4(2c^2 + 2d^2) &= \frac{24}{5}g^2 \\ 4(2d^2) + \frac{24}{5}k^2 &= \frac{24}{5}g^2 \\ 4(3d^2) + \frac{24}{5}k_4^2 &= \frac{24}{5}g^2 \\ 3(2a^2) + \frac{24}{5}k^2 &= \frac{24}{5}g^2 \\ 3(2b^2) + \frac{24}{5}k_4^2 &= \frac{24}{5}g^2 \\ 3k^2 + k_4^2 + \frac{21}{5}\lambda^2 &= \frac{24}{5}g^2. \end{aligned} \quad (38)$$

The solution to this set of equations has one free parameter. We choose it be  $k_4$ , in which case the solution is:

$$\begin{aligned} (a^2, b^2, c^2, d^2) &= \left( \frac{8}{15}g^2 + \frac{4}{15}k_4^2, \frac{8}{15}g^2 - \frac{4}{15}k_4^2, \frac{2}{15}g^2 + \frac{1}{5}k_4^2 \right) \\ (e^2, k^2, \lambda^2) &= \left( \frac{2}{5}g^2 - \frac{2}{5}k_4^2, \frac{1}{3}g^2 - \frac{1}{3}k_4^2, \frac{15}{7}g^2 \right). \end{aligned} \quad (39)$$

To eliminate this undetermined parameter  $k_4$  one needs to introduce an additional symmetry. A  $Z_2$  symmetry can set  $k_4 = 0$ , but if this  $Z_2$  commutes with  $S_4$ , it will also set some other parameters to be zero. We suspect a  $Z_2$  that does not commute with the  $S_4$  symmetry might set  $k_4$  equal to zero, while preserving the solution Eq. (39). We find the model phenomenologically interesting for this case. The mass matrix is for the up-type quarks is:

$$M^u = \sqrt{\frac{8}{15}} g \begin{pmatrix} \langle H_3 \rangle + \langle H_4 \rangle & 0 & \langle H_1 \rangle \\ 0 & \langle H_4 \rangle & \langle H_2 \rangle \\ \langle H_1 \rangle & \langle H_2 \rangle & -\langle H_3 \rangle + \langle H_4 \rangle \end{pmatrix}, \quad (40)$$

and a similar form for the down-type quarks.

To explain the mass hierarchy, we first set the  $(1, 1)$  entry of the mass matrices both in the up and the down sectors to be zero by choosing  $\langle H_3 \rangle$  and  $\langle \bar{H}_3 \rangle$  as:

$$\langle H_3 \rangle + \langle H_4 \rangle \sim 0, \quad \langle \bar{H}_3 \rangle + \langle \bar{H}_4 \rangle \sim 0.$$

Furthermore, we take  $\langle H_1 \rangle$  and  $\langle \bar{H}_1 \rangle$  to be smaller than  $\langle H_2 \rangle \sim \langle H_4 \rangle$ . One immediate observation from the structure is that the rotation between the second and the third generations is large. These large rotations from the up and the down sectors will cancel out. Let us define  $\frac{\langle H_2 \rangle}{\langle H_4 \rangle} = \sqrt{2}(1 + \delta^u)$ . In the limit  $\epsilon^u \equiv \frac{\langle H_1 \rangle}{\langle H_4 \rangle} \rightarrow 0$ , the rotation in the second and third generations is:

$$\begin{pmatrix} 1 & \sqrt{2}(1 + \delta^u) \\ \sqrt{2}(1 + \delta^u) & 2 \end{pmatrix}. \quad (41)$$

From this one finds

$$\frac{m_c}{m_t} = -\frac{4}{9}\delta^u, \quad (42)$$

where  $m_c$  and  $m_t$  are the masses of charm and top quarks. The rotation angle is:

$$\tan(2\theta_{23}^u) = 2\sqrt{2}(1 + \delta^u). \quad (43)$$

The large rotation angle will cancel out in  $V_{cb}$ , leaving only the smaller corrections proportional to  $\delta^{u,d}$ . The large rotation in the 2-3 space will induce an entry equal to  $\epsilon^u \sin \theta_{23}^u \langle H_4 \rangle$  in the  $(1,3)$  element. From this, we obtain the following relations for the quark mixing angles:

$$\begin{aligned} V_{cb} &= \frac{1}{2\sqrt{2}} \left| \frac{m_s}{m_b} \pm \frac{m_c}{m_t} \right| \\ V_{us} &= \left| \sqrt{\frac{m_d}{m_s}} \pm \sqrt{\frac{m_u}{m_c}} \right| \\ V_{ub} &= 2\sqrt{2} \left| \frac{\sqrt{m_d m_s}}{m_b} \pm \frac{\sqrt{m_u m_c}}{m_t} \right|. \end{aligned} \quad (44)$$

These are of the right order of magnitude, although in detail, the magnitude of  $V_{cb}$  is somewhat smaller than what is needed and  $V_{ub}$  is on the larger side. We consider this general agreement with experiments to be encouraging.

## 4 Concluding remarks

We have presented in this paper several models for quark masses and mixings in the context of finite  $SU(5)$  GUT. These theories are attractive candidates for an underlying theory, since the  $\beta$  functions for the gauge and Yukawa couplings vanish to all orders in perturbation theory. The requirements on the theory to be finite also leads to Yukawa-gauge unification, leading to a single coupling constant in the theory.

The models presented are based on non-Abelian discrete symmetry, which seem to be necessary to obtain isolated and non-degenerate solutions to the Yukawa couplings when expressed as power series in terms of the gauge coupling. We find it interesting that realistic quark masses and mixings can be generated in such a framework.

There are several open questions, many of which cannot be addressed until after finding a consistent quark mixing scheme. An important question finite theories should address is how to avoid rapid proton decay. Because all the Yukawa couplings, including those of the light generations, are order of  $g$ , color triplet exchange will generate a large amplitude for proton decay through  $d = 5$  operators [14]. This may simply be a technical problem associated with using  $SU(5)$  as the gauge group. One can envision other groups without the color triplets, although no realistic model of this type are known to us. Within finite  $SU(5)$ , there are ways to suppress the troublesome proton decay operators. For example, if the SUSY particle spectrum is such that the gauginos are light (of order 100 GeV), while the squarks are very heavy (of order 100 TeV or larger), the  $d = 5$  proton decay problem goes away. Although this choice may not be that attractive from the point of view of solving the gauge hierarchy problem, we emphasize that finiteness of the theory says nothing per se about the scale of SUSY breaking. A third alternative is to suppose that the masses of all the extra color triplets in the theory are much heavier than the GUT scale, even larger than the Planck scale.

In the framework of  $SU(5)$  finite GUT, the following question arises naturally: Is it possible to generate small neutrino masses? If right-handed neutrinos are introduced as  $SU(5)$  singlets, they can have no Yukawa couplings with the other fields, due to the demand of finiteness. We mention two possibilities to induce small neutrino masses. One is through bilinear  $R$ -parity violating terms of order the weak scale [20]. That does not contradict the requirements of finiteness. Another possibility is to make use of Planck suppressed higher dimensional operators, which can be constructed within finite  $SU(5)$ .

As we have shown in Sec. 2, within finite  $SU(5)$ , all four pairs of  $\mathbf{5} + \bar{\mathbf{5}}$  fields present in the theory must be Higgs-like. This is needed for achieving doublet-triplet splitting. If one pair were fermionic, the bad mass relations of  $SU(5)$ , viz.,  $m_s = m_\mu$  and  $m_d = m_e$  could have been corrected by terms such as  $\bar{\mathbf{5}}_i \mathbf{5}_i$  bilinear mass terms along with  $\bar{\mathbf{5}} \Sigma \mathbf{5}$  coupling. Since that is not possible, one has to rely on either Planck suppressed operators or finite gaugino diagrams to split the masses of leptons versus down type quarks [21]. Both possibilities appear to be viable.

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